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Real de Sitter and conformal algebras in terms of q-oscillators

Arkadiusz Kornakiewicz

Institute of Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland

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Abstract. We express the Cartan–Weyl basis of all quantum real $D=4$ de Sitter and $D=4$ conformal algebras in terms of q-deformed oscillators.

1. Introduction

Recently, several authors have described the realizations of the q-deformed Lie algebra $U_q(\hat{q})$ in terms of q-deformed oscillators [1–5]. In particular, the Drinfeld–Jimbo realizations of the Cartan–Chevaley basis of the q-deformed Lie algebras for the classical Cartan series A_N, B_N, C_N, D_N have been expressed by the q-oscillators firstly by Hayashi [6]. The following two problems arise:

(*a*) How to express the Cartan–Weyl basis of $U_q(\hat{q})$ in terms of q-oscillators.

(*b*) How to obtain the real forms of $U_q(\hat{q})$ by imposing the hermicity conditions for q-oscillators.

The first question has been answered for particular examples in [7]; recently, the construction has been presented for $U_q(\mathscr{U}(N))$ [8].

In this paper we shall consider mainly the problem of how to describe the real forms of quantum Lie algebras by q-oscillators. Because of the physical applications, we shall consider here examples of standard real forms of $U_q(sp(4;C))$ and $U_q(sl(4;C))$, classified in [9] and [10].

We recall that the standard real form is defined as a +-operation on the elements $a, b\in\mathscr{A}$, where $\mathscr{A}$ is a Hopf algebra (S is the antipode), with properties

$$(ab)^+=b^+a^+ \tag{1.1a}$$

$$(a\otimes b)^+=a^+\otimes b^+ \tag{1.1b}$$

$$S\circ+\circ S\circ+=1\,. \tag{1.1c}$$

Because the quantum oscillators in their algebraic formulation do not form the non-commutative Hopf algebra with algebra as well as co-algebra sectors, our embedding of $U_q(\hat{q})$ into the algebra of bilinears of q-oscillators describes only the algebra sector. One can therefore also look for the description of real forms defined with the help of the $\oplus$-operation (see [9]).

$$(ab)^\oplus=b^\oplus a^\oplus \tag{1.2a}$$

$$(a\otimes b)^\oplus=b^\oplus\otimes a^\oplus\,. \tag{1.2b}$$

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In general, one can also consider two $*$-operations, forming the non-standard real forms $*$ and $\odot$ [9], where $*$ is an automorphism of the algebra. However, in this paper we shall consider only the involutions which satisfy (1.1a) (or 1.2a).

It is interesting to observe that the quantum oscillators can be used for the description of the fundamental quantum spinors, describing the co-representations of quantum real $D=4$ de Sitter and quantum real $D=4$ conformal algebras. In such a way we obtain the q-deformation of the components of real $sp(4)$ spinors and $D=4$ twistors.

2. Antipode-extended Cartan–Weyl basis for $U_q(sl(4;C))$ and $U_q(sp(4;C))$

In order to obtain the q-oscillator realization of $U_q(sl(4;C))$ as well as $U_q(sp(4;C))$ we introduce holomorphic oscillators $a_i, b_j (i=1,\ldots,4$ for $U_q(sl(4;C))$; $i=1,2$ for $U_q(sp(4;C)))$ which obey the relations

$$\begin{aligned} &a_i b_j - q b_j a_i = q^{-N_i}\delta_{ij} \\ &[b_i, b_j] = 0 = [a_i, a_j] \end{aligned} \tag{2.1}$$

where N_i is the number operator satisfying

$$\begin{aligned} &[N_i, a_j] = -a_i\delta_{ij} \\ &[N_i, b_j] = b_i\delta_{ij}. \end{aligned} \tag{2.2}$$

(a) $U_q(sl(4;C))$. The Cartan–Weyl basis for $U_q(sl(4;C))$ in terms of these complex oscillators is given by

$$\begin{aligned} &e_1 = b_1 a_2 && e_{-1} = a_1 b_2 \\ &e_2 = b_2 a_3 && e_{-2} = a_2 b_3 \\ &e_3 = b_3 a_4 && e_{-3} = a_3 b_4 \\ &e_4 = b_1 a_3 q^{-2N_2} && e_{-4} = a_1 b_3 q^{2N_2} \\ &e_5 = b_2 a_4 q^{-2N_3} && e_{-5} = a_2 b_4 q^{2N_3} \\ &e_6 = b_1 a_4 q^{-2(N_2+N_3)} && e_{-6} = a_1 b_4 q^{2(N_2+N_3)} \\ &h_1 = N_1 - N_2 && h_4 = j_1 + h_2 = N_1 - N_3 \\ &h_2 = N_2 - N_3 && h_5 = h_2 + h_3 = N_2 - N_4 \\ &h_3 = N_3 - N_4 && h_6 = h_1 + h_5 = N_1 - N_4 \end{aligned} \tag{2.3}$$

where h_j describe the Cartan subalgebra, e_j, e_{-j} $(j=1,2,3)$ are the generators corresponding to simple roots, and e_j, e_{-j} $(j=4,5,6)$ are the generators corresponding to non-simple roots defined as (our definitions differ from that in [9] and [10] by replacement of q by q^2)

$$\begin{aligned} &e_4 \equiv [e_1, e_2]_{q^2} && e_{-4} \equiv [e_{-2}, e_{-1}]_{q^{-2}} \\ &e_5 \equiv [e_2, e_3]_{q^2} && e_{-5} \equiv [e_{-3}, e_{-2}]_{q^{-2}} \\ &e_6 \equiv [e_1, e_5]_{q^2} && e_{-6} \equiv [e_{-5}, e_{-1}]_{q^{-2}} \end{aligned} \tag{2.4a}$$

where $[A, B]_x \equiv AB - xBA$.

In order to examine the real forms in the case when the q-parameter is real we should introduce the so-called antipode-extended Cartan–Weyl basis defined by

$$\begin{aligned}
&\tilde{e}_4 \equiv [e_2, e_1]_{q^2} &\quad &\tilde{e}_{-4} \equiv [e_{-1}, e_{-2}]_{q^{2}} \\
&\tilde{e}_5 \equiv [e_3, e_2]_{q^2} &\quad &\tilde{e}_{-5} \equiv [e_{-2}, e_{-3}]_{q^{2}} \\
&\tilde{e}_6 \equiv [\tilde{e}_5, e_1]_{q^2} &\quad &\tilde{e}_{-6} \equiv [e_{-1}, \tilde{e}_{-5}]_{q^{2}}
\end{aligned} \tag{2.4b}$$

which assume the following forms:

$$\begin{aligned}
&\tilde{e}_4 = -q^2 b_1 a_3 q^{2N_2} &\quad &\tilde{e}_{-4} = -q^{-2} a_1 b_3 q^{-2N_2} \\
&\tilde{e}_5 = -q^2 b_2 a_4 q^{2N_3} &\quad &\tilde{e}_{-5} = -q^{-2} a_2 b_4 q^{-2N_3} \\
&\tilde{e}_6 = q^4 b_1 a_4 q^{2(N_2+N_3)} &\quad &\tilde{e}_{-6} = q^{-4} a_1 b_4 q^{-2(N_2+N_3)}.
\end{aligned} \tag{2.5}$$

(b) $U_q(sp(4; C))$. In an analogous way we introduce the basis for $U_q(sp(4; C))$:

$$\begin{aligned}
&e_1 = \frac{1}{[2]^{1/2}} b_1 a_2 &\quad &e_{-1} = \frac{1}{[2]^{1/2}} a_1 b_2 \\
&e_2 = \frac{i}{[2]} b_2^2 &\quad &e_{-2} = \frac{i}{[2]} a_2^2 \\
&e_3 = [e_1, e_2]_{q^2} = \frac{i}{[2]^{1/2}} b_1 b_2 q^{-N_2} &\quad &e_{-3} = \frac{i}{[2]^{1/2}} q^{N_2} a_2 a_1 \\
&e_4 = [e_1, e_3] = \frac{i}{[2]} b_1^2 q^{-2N_2} &\quad &e_{-4} = \frac{i}{[2]} a_1^2 q^{2N_2} \\
&\tilde{e}_3 = -\frac{i}{[2]^{1/2}} q^2 b_2 b_1 q^{N_2} &\quad &\tilde{e}_{-3} = -\frac{i}{[2]^{1/2}} q^{-2} q^{-N_2} a_2 a_1 \\
&\tilde{e}_4 = \frac{i}{[2]} q^2 b_1^2 q^{2N_2} &\quad &\tilde{e}_{-4} = \frac{i}{[2]} q^{-2} a_1^2 q^{-2N_2}
\end{aligned} \tag{2.6}$$

The Cartan subalgebra takes the form

$$\begin{aligned}
&h_1 = \tfrac{1}{2}(N_1 - N_2) \\
&h_2 = N_2 + \tfrac{1}{2} \\
&h_3 = h_1 + h_2 = \tfrac{1}{2}(N_1 + N_2 + 1) \\
&h_4 = 2h_1 + h_2 = N_1 + \tfrac{1}{2}
\end{aligned} \tag{2.7}$$

where

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}.$$

3. Oscillator realizations of real algebra $U_q(O(3, 2))$

Appropriate reality conditions should be imposed on the Cartan–Weyl basis of $U_q(sp(4; C))$ in order to describe the real quantum realization of $U_q(O(3, 2))$. We shall recall the real forms of $U_q(O(3,2))$ and point out which real forms can be represented by q-oscillators.

(i) '+' involution, which is the algebra anti-automorphism and co-algebra automorphism:

$$\begin{aligned} &h_i^+ = -h_i && (|q|=1;\ \varepsilon, \lambda = \pm 1) \\ &e_{\pm 1}^+ = \lambda e_{\pm 1} && e_{\pm 3}^+ = -\lambda\varepsilon q^{\mp 2} e_{\pm 3} \\ &e_{\pm 2}^+ = \varepsilon e_{\pm 2} && e_{\pm 4}^+ = \varepsilon q^{\mp 2} e_{\pm 4}. \end{aligned} \tag{3.1a}$$

The above conditions can be expressed in terms of oscillators, which gives the following oscillator conditions ($j=1,2$):

(a) $(\varepsilon, \lambda) = (-1, -1)$

$$\begin{aligned} &a_1^+ = \pm \mathrm{i} b_2 && a_j^+ = \pm a_j \\ &a_2^+ = \pm \mathrm{i} b_1 && b_j^+ = \pm b_j \end{aligned} \tag{3.1b}$$

(b) $(\varepsilon, \lambda) = (-1, -1)$

$$\begin{aligned} &a_1^+ = \pm b_2 && a_j^+ = \pm(-1)^j a_j \\ &a_2^+ = \pm b_1 && b_j^+ = \pm(-1)^j b_j \end{aligned} \tag{3.1c}$$

(c) $(\varepsilon, \lambda) = (1, 1)$

$$\begin{aligned} &a_1^+ = \pm b_2 && a_j^+ = \pm \mathrm{i}(-1)^j a_j \\ &a_2^+ = \pm b_1 && b_j^+ = \pm \mathrm{i}(-1)^j b_j \end{aligned} \tag{3.1d}$$

(d) $(\varepsilon, \lambda) = (1, -1)$

$$\begin{aligned} &a_1^+ = \pm \mathrm{i} b_2 && a_j^+ = \pm(-1)^j a_j \\ &a_2^+ = \pm \mathrm{i} b_1 && b_j^+ = \pm(-1)^j b_j. \end{aligned} \tag{3.1e}$$

In all cases (a)–(d) the number operators satisfy the condition

$$N_j^+ = -(N_j + 1).$$

(ii) '$\oplus$' involution, which is the anti-automorphism in both the algebra and co-algebra sectors:

$$\begin{aligned} &h_i^{\oplus} = h_i && (|q|=1) \\ &e_{\pm 1}^{\oplus} = \lambda e_{\mp 1} && e_{\pm 3}^{\oplus 4} = \lambda\varepsilon e_{\mp 3} \\ &e_{\pm 2}^{\oplus} = \varepsilon e_{\mp 2} && e_{\pm 4}^{\oplus} = \varepsilon e_{\mp 4}. \end{aligned} \tag{3.2a}$$

In terms of oscillators these relations can be realized if

$$\varepsilon = -1 \qquad \lambda = -1$$

or

$$\varepsilon = -1 \qquad \lambda = 1.$$

The following two real forms of $U_q(O(3,2))$ can be given:

(a) $(\varepsilon, \lambda) = (-1, -1)$ (3.2b)

$$\begin{aligned} &a_1^{\oplus} = \pm b_1 \\ &a_2^{\oplus} = \mp b_2 \end{aligned}$$

(b) $(\varepsilon, \lambda) = (-1, 1)$

$$\begin{aligned} &a_1^{\oplus} = \pm b_1 \\ &a_2^{\oplus} = \pm b_2. \end{aligned} \tag{3.2c}$$

The realization (3.2c) has been proposed in [11].

(iii) '$\oplus$' involution, which is the anti-automorphism in both the algebra and co-algebra sectors:

$$
\begin{aligned}
&h_i^{\oplus}=h_i && (g\in\mathbb{R})\\
&e_{\pm 1}^{\oplus}=\lambda e_{\mp 1} \qquad e_{\pm 3}^{\oplus}=-\varepsilon\lambda q^{\pm 2}\tilde{e}_{\mp 3}\\
&e_{\pm 2}^{\oplus}=\varepsilon e_{\mp 2} \qquad e_{\pm 4}^{\oplus}=\varepsilon q^{\pm 2}\tilde{e}_{\mp 4}.
\end{aligned} \tag{3.3}
$$

In terms of q-oscillators one can express only two realizations of (3.3) if $\varepsilon=-1$, $\lambda=1$ or $\varepsilon=-1$, $\lambda=-1$:

(a) $(\varepsilon, \lambda)=(-1, -1)$

$$a_1^{\oplus}=\pm b_1$$

$$a_2^{\oplus}=\mp b_2$$

(b) $(\varepsilon, \lambda)=(-1, 1)$

$$a_1^{\oplus}=\pm b_1$$

$$a_2^{\oplus}=\pm b_2.$$

In the case of all the equations (3.2) and (3.3) the number operators satisfy

$$N_j^{+}=N_j \qquad j=1,2.$$

We see that in the case of two oscillators we obtain only the realizations corresponding to $U_q(O(3, 2))$. The oscillator realizations of $U_q(O(4, 1))$ and $U_q(O(5))$ require at least four oscillators.

4. Oscillator realization of real conformal algebra $U_q(O(4, 2))$

First we recall the complete list of standard real forms of $U_q(sl(4; C))$ according to [10]. For the conformal case we do not consider the $\oplus$-involution. The involutions $\Phi_k (k=1, \ldots, 5)$ are completely described by its action on simple roots:

$$
\begin{aligned}
&\Phi_1(h_j)=-h_{4-j} && \Phi_1(e_{\pm j})=e_{\pm(4-j)} && (|q|=1)\\
&\Phi_2(h_j)=-h_j && \Phi_2(e_{\pm j})=e_{\pm j} && (|q|=1)\\
&\Phi_3(h_j)=h_{4-j} && \Phi_3(e_{\pm j})=e_{\mp(4-j)} && (q\in\mathbb{R})\\
&\Phi_4(h_j)=h_{4-j} && \Phi_4(e_{\pm j})=(-1)^{\delta_{j2}}e_{\mp(4-j)} && (q\in\mathbb{R})\\
&\Phi_5(h_j)=h_j && \Phi_5(e_{\pm j})=\varepsilon_j e_{\mp j} && (\varepsilon_j=\pm 1, q\in\mathbb{R}).
\end{aligned}
$$

Further, we shall restrict our considerations to the case of the $D=4$ conformal algebra $U_q(O(4, 2))$. Only by means of Φ_1 and Φ_5 can we construct a deformation of the $D=4$ conformal algebra, and in both cases we should use the antipode-extended basis for $U_q(sl(4; C))$. It appears that all four standard real forms listed in [10] of $U_q(O(4, 2))$

Table 1. The oscillator realization for real forms of $U_q(O(4,2))$, where for Φ_1 the number operators satisfy $N_j^{\dagger}=N_{4-(j-1)}$ while for Φ_5 they satisfy $N_j^{\dagger}=N_j$. In all the relations $j=1,2,3,4$.

Type of involution	Conditions in terms of q-oscillators			
Φ_1	$a_1^{\dagger}=\pm b_4$	$a_2^{\dagger}=\pm b_3$	$a_3^{\dagger}=\pm b_2$	$a_4^{\dagger}=\pm b_1$
$\Phi_5;(\varepsilon,\varepsilon_2)=(1,-1)$	$a_1^{\dagger}=\pm b_1$	$a_2^{\dagger}=\pm b_2$	$a_3^{\dagger}=\mp b_3$	$a_4^{\dagger}=\mp b_4$
$\Phi_5;(\varepsilon,\varepsilon_2)=(-1,1)$	$a_1^{\dagger}=\pm b_1$	$a_2^{\dagger}=\pm b_2$	$a_3^{\dagger}=\pm b_3$	$a_4^{\dagger}=\pm b_4$
$\Phi_5;(\varepsilon,\varepsilon_2)=(-1,-1)$	$a_1^{\dagger}=\pm b_1$	$a_2^{\dagger}=\mp b_2$	$a_3^{\dagger}=\pm b_3$	$a_4^{\dagger}=\mp b_4$

can be represented by the q-oscillators. We describe the reality conditions for q-oscillators in table 1 ($\varepsilon\equiv\varepsilon_1=\varepsilon_3$). The reality conditions for q-oscillators in table 1 describe different forms of the reality conditions for quantum twistors.

5. Covariance relations for $U_q(sp(4;C))$ and $U_q(sl(4;C))$

We introduce below the q-commutation relations between generators of quantum algebra $U_q(sp(4;C))$ ($U_q(sl(4;C))$) and q-deformed oscillators. Further, we assume the following notation: e_A are generators of quantum algebra $U_q(sp(4;C))$ or $U_q(sl(4;C))$ ($A=1,\ldots,4$ for $U_q(sp(4;C))$; $A=1,\ldots,6$ for $U_q(sl(4;C))$); $a_A=(a_i,b_i)$ are complex oscillators expressing e_A in terms of the q-oscillators (2.3), (2.6) ($i=1.2$ for $U_q(sp(4;C))$; $i=1,\ldots,=4$ for $U_q(sl(4;C))$).

(a) $U_q(sp(4;C))$.

$$[e_A,a_B]_{g(A,B)}=\begin{pmatrix} -a_2q^{N_1+1}\dfrac{1}{[2]^{1/2}} & 0 & 0 & b_1q^{-N_2}\dfrac{1}{[2]^{1/2}} \\ 0 & \mathrm{i}b_2q^{N_2}(1-q^4) & 0 & 0 \\ -\dfrac{\mathrm{i}}{[2]^{1/2}}b_2q^{-N_2+N_1+1} & -\dfrac{\mathrm{i}}{[2]^{1/2}}b_1q^{-2N_2} & 0 & 0 \\ \mathrm{i}b_1q^{N_1-2N_2}(1-q^4) & 0 & 0 & 0 \end{pmatrix} \tag{5.1a}$$

where

$$g_{(A,B)}=\begin{pmatrix} q & 1 & 1 & q \\ 1 & q^2 & 1 & 1 \\ q & 1 & 1 & q^{-1} \\ q^2 & q^2 & 1 & q^{-2} \end{pmatrix}. \tag{5.1b}$$

If we add to (5.1a) the conjugated relations ($e_{\pm i}\to e_{\mp i}$, $q\to q^{-1}$), we obtain the complete set of covariance relations.

(b) $U_q(sl(4;C))$. We have the following relations for $U_q(sl(4;C))$:

$$[c_A, a_B]_{g(A,B)} = \begin{pmatrix} -a_2q^{2(N_1+1)} & 0 & 0 & 0 & 0 & b_1q^{-2N_2} & 0 & 0 \\ 0 & -a_3q^{2(N_2+1)} & 0 & 0 & 0 & 0 & b_2q^{-2N_3} & 0 \\ 0 & 0 & -a_4q^{2(N_3+1)} & 0 & 0 & 0 & 0 & b_3q^{-2N_4} \\ -a_3q^{2(N_1-N_2+1)} & 0 & 0 & 0 & 0 & 0 & b_1q^{-2(N_2+N_3)} & 0 \\ 0 & -a_4q^{2(-N_3+N_2+1)} & 0 & 0 & 0 & 0 & 0 & b_2q^{-2(N_3+N_4)} \\ -a_4q^{2(N_1-N_2-N_3+1)} & 0 & 0 & 0 & 0 & 0 & 0 & b_1q^{-2(N_2+N_3+N_4)} \end{pmatrix}. \tag{5.2a}$$

where

$$g(A,B) = \begin{pmatrix} q^2 & 1 & 1 & 1 & 1 & q^2 & 1 & 1 \\ 1 & q^2 & 1 & 1 & 1 & 1 & q^2 & 1 \\ 1 & 1 & q^2 & 1 & 1 & 1 & 1 & q^2 \\ q^2 & q^2 & 1 & 1 & 1 & q^{-2} & q^2 & 1 \\ 1 & q^2 & q^2 & 1 & 1 & 1 & q^{-2} & q^2 \\ q^2 & q^2 & q^2 & 1 & 1 & q^{-2} & q^{-2} & q^2 \end{pmatrix}. \tag{5.2b}$$

The collection of four oscillators $a_B (B = 1, 2, 3, 4)$ describes the non-commuting coordinates of complexified twistor space.

6. Final remarks

In this paper we have investigated the oscillator realizations of q-deformed de Sitter and conformal algebras. It appears that one can obtain the realizations for all eight real forms of $U_q(O(3,2))$ and four real forms of $U_q(O(4,2))$. Using the minimal number of q-oscillators ($n=2$ for $U_q(sp(4,C))$; $n=4$ for $U_q(sl(4;C))$ we were not able to obtain the oscillator realization of quantum $O(4,1)$ and $O(5)$ algebras which have ($q=1$) fundamental quaternionic realizations.

It would be very interesting to extend this formalism to the superconformal quantum deformations. In such a case one has to use bosonic as well as fermionic q-deformed oscillators.

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References

[1] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873–8
[2] Kulish P and Damaskinsky E 1990 *J. Phys. A: Math. Gen.* **23** L415
[3] Macfarlane G 1989 *J. Phys. A: Math. Gen.* **22** 4581
[4] Sun C P and Fu M 1984 *J. Math. Phys. A: Math. Gen.* **22** 2983

[5] Floreanini R 1990 *Preprint* UCLA 90/TEP 219
[6] Hayashi T 1990 *Commun. Math. Phys.* **127** 129–44
[7] Burdik Č, Černy L and Navratil O 1992 *Preprint* PRA-HEP-92/16
[8] Quesne C 1993 *J. Phys. A: Math. Gen.* **26** 357–72
[9] Lukierski J, Nowicki A and Ruegg H 1991 *Phys. Lett.* **271B** 321–8
[10] Lukierski J, Nowicki A and Sobczyk J 1993 *J. Phys. A: Math. Gen.* **26** 4047
[11] Chaichian M, de Azcaraga J A, Prešnajder P. and Rodenas F 1992 *Phys. Rev.* B **291** 411